Symmetries of second-order PDEs and conformal Killing vectors

Michael Tsamparlis[1] **and Andronikos Paliathanasis**[2,3]

Abstract. We study the Lie point symmetries of a general class of partial differential equations (PDE) of second order. An equation from this class naturally defines a second-order symmetric tensor (metric). In the case the PDE is linear on the first derivatives we show that the Lie point symmetries are given by the conformal algebra of the metric modulo a constraint involving the linear part of the PDE. Important elements in this class are the Klein–Gordon equation and the Laplace equation. We apply the general results and determine the Lie point symmetries of these equations in various general classes of Riemannian spaces. Finally we study the type II hidden symmetries of the wave equation in a Riemannian space with a Lorenzian metric.

[1] Department of Physics, Section of Astronomy, Astrophysics and Mechanics, University of Athens, Panepistemiopolis, Athens 157 83, Greece
[2] Dipartimento di Fisica, Università di Napoli "Federico II" Complesso universitario Monte S. Angelo Via, Cintia - 80126 Napoli, Italy
[3] INFN - Istituto Nazionale di Fisica Nucleare sezione di Napoli Complesso universitario Monte S. Angelo Via Cintia - 80126 Napoli, Italy

E-mail: `mtsampa@phys.uoa.gr, paliathanasis@na.infn.it`

1. Introduction

In theoretical physics one has two main tools to study the properties of evolution of dynamical systems (a) Symmetries of the equations of motion and (b) Collineations (symmetries) of the background space, where evolution takes place. It is well known that both these tools have the following common characteristics:

1) they form a Lie algebra;

2) they do not fix uniquely either the dynamical system or the space.

The natural question to be to asked is if these two algebras are related and in what way. Equivalently, one may state the question as follows:

To what degree and how the space modulates the evolution of dynamical systems in it? That is, a dynamical system is free to evolve at will in a given space or it is constrained to do so by the very symmetry structure of the space?

This question has been answered many years ago by the Theory of Relativity with the Equivalence Principle, that is, the requirement that free motion in a given gravitational field occurs along the geodesics of the space. However as obvious as this point of view may appear to be it is not easy to comprehend and accept! So let us give a precise formulation now[1].

[1] This point has been raised during an illuminating discussion with Prof P G Leach in 2000 in Athens while we were driving to the Poseidon Temple in Cape Sounion.

In a Riemannian space the affinely parameterized geodesics are determined uniquely by the metric. The geodesics are a set of homogeneous ordinary differential equations (ODE) linear in the highest order term and quadratically non-linear in the first order terms. A system of such ODEs is characterized (not fully) by its Lie point symmetries. On the other hand a metric is characterized (again not fully) by its collineations. Therefore it is reasonable one to expect that the Lie point symmetries of the system of geodesic equations of a metric will be closely related with the collineations of the metric. That such a relation exists it is easy to see by the following simple example. Consider on the Euclidian plane a family of straight lines parallel to the x-axis. These curves can be considered either as the integral curves of the ODE $\frac{d^2y}{dx^2}=0$ or as the geodesics of the Euclidian metric dx^2+dy^2. Subsequently consider a symmetry operation defined by a reshuffling of these lines without preserving necessarily their parametrization. According to the first interpretation this symmetry operation is a Lie symmetry of the ODE $\frac{d^2y}{dx^2}=0$ and according to the second interpretation it is a (special) projective symmetry of the Euclidian two-dimensional space.

What has been said for a Riemannian space can be generalized to an affine space in which there is only a linear connection. In this case the geodesics are called autoparallels (or paths) and they comprise again a system of ODEs linear in the highest order term and quadratically non-linear in the first order terms. In this case one is looking for relations between the Lie point symmetries of the autoparallels and the projective collineations of the connection.

A Lie point symmetry of an ordinary differential equation (ODE) is a point transformation in the space of variables which preserves the set of solutions of the ODE [1–3]. If we look at these solutions as curves in the space of variables, then we may equivalently consider a Lie point symmetry as a point transformation which preserves the set of the solution curves. Applying this observation to the geodesic curves in a Riemannian (affine) space, we infer that the Lie point symmetries of the geodesic equations in any Riemannian space are the automorphisms which preserve the set of these curves. However we know from Differential Geometry that the point transformations of a Riemannian (affine) space which preserve the set of geodesics are the projective transformations. Therefore it is reasonable to expect a correspondence between the Lie point symmetries of the geodesic equations and the projective algebra of the metric defining the geodesics.

The equation of geodesics in an arbitrary coordinate frame is a second-order ODE of the form

$$\ddot{x}^i+\Gamma^i_{jk}\dot{x}^j\dot{x}^k+F^i(x,\dot{x})=0, \tag{1}$$

where $F^i(x,\dot{x})$ are arbitrary functions of their arguments and the functions Γ^i_{jk} are the connection coefficients of the space. Equivalently equation (1) is also the equation of motion of a dynamical system moving in a Riemannian (affine) space under the action of a velocity dependent force. According to the above argument we expect that the Lie point symmetries of the ODE (1) for given functions $F^i(x,\dot{x})$ will be related with the collineations of the metric. As it will be shown in this case the Lie symmetries of (1) determine a subalgebra of the special projective algebra of the space. The specific subalgebra is selected by means of certain constraint conditions involving geometric quantities of the space and the function $F(x^i,\dot{x}^j)$ [4–9].

The determination of the Lie point symmetries of a given system of ODEs consists of two steps: (a) determination of the conditions which the components of the Lie symmetry vector must satisfy and (b) solution of the system of these conditions. Step (a) is formal and it is outlined, e.g., in [1–3]. The second step is the key one and, for example, in higher dimensions, where one has a large number of equations, the solution can be quite involved and perhaps impossible by algebraic computing. However, if one expresses the system of Lie symmetry conditions of (1) in terms of collineation (i.e. symmetry) conditions of the metric, then the determination of Lie point symmetries is transferred to the geometric problem of determining

the special projective group of the metric [9]. In this field there is a significant amount of knowledge from Differential Geometry waiting to be used. Indeed the projective symmetries are already known for many spaces or they can be determined by existing general theorems. For example the projective algebra and all its subalgebras are known for the important case of spaces of constant curvature [10] and in particular for the flat spaces. This implies that, the Lie symmetries of the Newtonian dynamical systems as well as those of Special Relativity can be determined by simple differentiation from the known projective algebra of these spaces!

What has been said for the Lie point symmetries of (1) applies also to Noether point symmetries (provided (1) follows from a Lagrangian). The Noether point symmetries are Lie point symmetries which satisfy the additional constraint

$$X^{[1]}L + L\frac{d\xi}{dt} = \frac{df}{dt}. \tag{2}$$

The Noether point symmetries form a subalgebra of the Lie point symmetry algebra. In accordance with the above this implies that the Noether point symmetries will be related with a subalgebra of the special projection algebra of the space where 'motion' occurs. As it has been shown this subalgebra is the homothetic algebra of the space [9]. It is well known that to each Noether point symmetry it is associated a conserved current (i.e. a Noether first integral). This leads to the important conclusion that the (standard) conserved quantities of a dynamical system depend *on the space it moves and the type of force* $F(x^i)$ *which modulates the motion.* In particular in 'free fall', that is when $F(x^i) = 0$, the orbits are affinely parametrized geodesics and the *geometry* of the space is the sole factor which determines the conserved quantities of motion. This conclusion is by no means trivial and means that the space where motion occurs is not a simple carrier of the motion but it is the major modulator of the evolution of a dynamical system. In other words there is a strong and deep relation between Geometry of the space and Physics (motion) in that space!

The above scenario can be generalized to partial differential equations (PDEs). Obviously, in this case a global answer is not possible. However, it can be shown that for many interesting PDEs the Lie point symmetries are indeed obtained from the collineations of the metric. Pioneering work in this direction is the work of Ibragimov [1]. Recently, Bozhkov et al. [11] studied the Lie and the point Noether symmetries of the Poisson equation and have shown that the Lie symmetries of the Poisson PDE are generated from the conformal algebra of the metric. This result can be generalized and it has been shown [12] that for a general class of PDEs of second order in an n-dimensional Riemannian space, there is a close relation between the Lie point symmetries and the conformal algebra of the space. Examples of such PDEs include some important equations as: the heat equation, the Klein–Gordon equation, the Laplace equation, the Schrödinger equation and others [12–15]. In what follows we discuss in a rather systematic way the aforementioned ideas. The plan of the paper is as follows.

In section 2 we give the basic definitions and properties of the collineations of space times and the Lie symmetries of DEs. In section 3 we study the Lie symmetries of a generic family of second-order partial differential equations and we prove that when the second-order partial differential equation is linear on the derivatives, the corresponding Lie symmetries are generated by the conformal algebra of the underlying geometry. Furthermore, in sections 4 and 5 we apply these general result in order to determine the general form of the Lie symmetry vector of the Poisson equation, of the Klein–Gordon equation and of the Laplace equation. In section 6 we study the application of the conformal Killing vectors in the Laplace equation is some special Riemannian spaces and we study the origin of the type II hidden symmetries. Finally in section 7 we apply the previous results in the case of the Laplace equation in the 1+3 wave equation and in Bianchi I spacetimes.

2. Preliminaries

In this section we give the basic definitions and properties of the collineations of spacetimes and of the point symmetries of differential equations.

2.1. Collineations of Riemannian spaces

A collineation in a Riemannian space of dimension n is a vector field $\mathbf{X}$ which satisfies an equation of the form $\mathcal{L}_X\mathbf{A}=\mathbf{B}$, where $\mathcal{L}_X$ is the Lie derivative with respect to the vector field $\mathbf{X}$, $\mathbf{A}$ is a geometric object (not necessary a tensor field) defined in terms of the metric and its derivatives, and $\mathbf{B}$ is an arbitrary tensor field with the same tensor indices as the geometric object $\mathbf{A}$. The classification of the collineations of Riemannian manifolds can be found in [18]. In the following we are interested in the collineations of the metric tensor, i.e. $\mathbf{A}=g_{ij}$ of the Riemannian space.

A vector field $\mathbf{X}$ is a conformal Killing vector (CKV) of g_{ij} if the following condition holds[2]:

$$\mathcal{L}_{\mathbf{X}}g_{ij}=2\psi\left(x^k\right)g_{ij},$$

where $\psi\left(x^k\right)=\frac{1}{n}\mathbf{X}^i_{;i}$. In case $\psi_{;ij}=0$, $\mathbf{X}$ is called special CKV (sp.CKV), if $\psi\left(x^k\right)=$ constant, the vector field X is called homothetic (HV) and if $\psi\left(x^k\right)=0$, the field $\mathbf{X}$ is called a Killing vector (KV).

The CKVs of the metric g_{ij} form a Lie algebra which is called the conformal algebra (CA) of the metric g_{ij} (CKVs). The conformal algebra contains two subalgebras, the Homothetic algebra (HA) and the Killing algebra (KA). These algebras are related as $KA\subseteq HA\subseteq CA$.

Two metrics g_{ij} and $\bar{g}_{ij}$ are conformally related if there exists a function $N^2\left(x^k\right)$ such as $\bar{g}_{ij}=N^2\left(x^k\right)g_{ij}$. If $\mathbf{X}$ is a CKV of the metric $\bar{g}_{ij}$ so that $L_{\mathbf{X}}\bar{g}_{ij}=2\bar{\psi}\bar{g}_{ij}$, then $\mathbf{X}$ is also a CKV of the metric g_{ij}, that is $L_{\mathbf{X}}g_{ij}=2\psi g_{ij}$ with conformal factor $\psi\left(x^k\right)$; the two conformal factors are related as follows

$$\psi=\bar{\psi}N^2-NN^i_{,i}\mathbf{X}^{,i}.$$

The last relation implies that two conformally related metrics have the same conformal algebra, but with different subalgebras; that is, a KV for one may be proper CKV for the other.

A special class of conformally related spaces are the conformally flat spaces. A space V^n is conformally flat if the metric g_{ij} of V^n satisfies the relation $g_{ij}=N^2s_{ij}$, where s_{ij} is the metric of a flat space which has the same signature as g_{ij}. The maximal dimension of the conformal algebra of a n-dimensional metric ($n>2$) is $\frac{1}{2}\left(n+1\right)\left(n+2\right)$ and in that case the space is conformally flat. Moreover, if the conformally flat space V^n admits a $\frac{1}{2}n\left(n+1\right)$-dimensional Killing algebra then V^n is a space of constant curvature and admits a proper HV if and only if the space is flat.

Furthermore, if for a CKV $\mathbf{X}$ of the Riemannian space V^n_G the condition $\mathbf{X}_{[i;j]}=0$ holds, i.e. $\mathbf{X}_{i;j}=\psi\left(x^k\right)g_{ij}$, then the CKV will be called gradient CKV. In this case there exists a coordinate system in which the line element of the metric which defines the Riemannian space V^n_G is

$$ds^2=dx^n+f^2\left(x^n\right)h^{AB}\left(x^A\right)dx^Adx^B,$$

where $A,B=1,2,\ldots,n-1$ [16]. In these coordinates the gradient CKV is $\mathbf{X}=f\left(x^n\right)\partial_{x^n}$ with conformal factor $\psi=f_{,x^n}$. In the case when $f\left(x^n\right)=x^n$, we have $\psi=1$; hence, $\mathbf{X}$ becomes gradient HV and if $f\left(x^n\right)=f_0$, $\mathbf{X}$ becomes gradient KV.

[2] In the following we use the Einstein notation.

2.2. Point symmetries of differential equations

A partial differential equation (PDE) is defined by a function $H = H(x^i, u^A, u^A_{,i}, u^A_{,ij})$ in the jet space $\bar{B}_{\bar{M}}$, where x^i are the independent variables and u^A are the dependent variables. The infinitesimal point transformation

$$\bar{x}^i = x^i + \varepsilon \xi^i \left(x^k, u^B\right), \tag{3}$$

$$\bar{u}^A = \bar{u}^A + \varepsilon \eta^A \left(x^k, u^B\right), \tag{4}$$

has the infinitesimal symmetry generator

$$\mathbf{X} = \xi^i \left(x^k, u^B\right) \partial_{x^i} + \eta^A \left(x^k, u^B\right) \partial_{u^A}. \tag{5}$$

The generator $\mathbf{X}$ of the infinitesimal transformation (3), (4) is called a Lie point symmetry of the PDE H if there exists a function λ such that the following condition holds [1, 2]

$$\mathbf{X}^{[n]}(H) = \lambda H, \quad \bmod H = 0, \tag{6}$$

where

$$\mathbf{X}^{[n]} = \mathbf{X} + \eta^A_i \partial_{\dot{x}^i} + \eta^A_{ij} \partial_{u^A_{ij\ldots i_n}} + \cdots + \eta^A_{i_1 i_2 \ldots i_n} \partial_{u^A_{i_1 i_2 \ldots i_n}} \tag{7}$$

is the n-th prolongation of $\mathbf{X}$ and

$$\eta^A_i = \eta^A_{,i} + u^B_{,i} \eta^A_{,B} - \xi^j_{,i} u^A_{,j} - u^A_{,i} u^B_{,j} \xi^j_{,B} \tag{8}$$

with

$$\begin{aligned} \eta^A_{ij} = {} & \eta^A_{,ij} + 2\eta^A_{,B(i} u^B_{,j)} - \xi^k_{,ij} u^A_{,k} + \eta^A_{,BC} u^B_{,i} u^C_{,j} - 2\xi^k_{,(i|B|} u^B_{j)} u^A_{,k} \\ & - \xi^k_{,BC} u^B_{,i} u^C_{,j} u^A_{,k} + \eta^A_{,B} u^B_{,ij} - 2\xi^k_{,(j} u^A_{,i)k} - \xi^k_{,B} \left(u^A_{,k} u^B_{,ij} + 2 u^B_{(,j} u^A_{,i)k} \right). \end{aligned} \tag{9}$$

Lie symmetries of differential equations can be used in order to determine invariant solutions or transform solutions into solutions [3]. From condition (6) one defines the Lagrange system

$$\frac{dx^i}{\xi^i} = \frac{du}{\eta} = \frac{du_i}{\eta_{[i]}} = \cdots = \frac{du_{ij\ldots i_n}}{\eta_{[ij\ldots i_n]}} \tag{10}$$

whose solution provides the characteristic functions

$$W^{[0]}\left(x^k, u\right), \quad W^{[1]i}\left(x^k, u, u_{,i}\right), \quad \ldots, \quad W^{[n]}\left(x^k, u, u_{,i}, \ldots, u_{,ij\ldots i_n}\right).$$

The solution $W^{[n]}$ of the Lagrange system (10) is called the n-th order invariant of the Lie symmetry vector (5) and holds $X^{[n]} W^{[n]} = 0$.

The application of a Lie symmetry to a PDE H leads to a new differential equation $\bar{H}$ which is different from H and it is possible that it admits Lie symmetries which are not Lie symmetries of H. These Lie point symmetries are called Type II hidden symmetries. It has been shown in [17] that if X_1, X_2 are Lie point symmetries of the original PDE with commutator $[X_1, X_2] = cX_1$ where c is a constant, then reduction by X_2 results in X_1 being a point symmetry of the reduced PDE $\bar{H}$, while reduction by X_1 results in a PDE $\bar{H}$ for which X_2 is not a Lie point symmetry.

In the following section we study the Lie point symmetries of a general type of second-order PDEs.

3. Lie symmetries of second-order PDEs and CKVs

It is interesting to examine if the close relation of the Lie and the Noether point symmetries of the second-order ODEs of the form (1) with the collineations of the metric is possible to be carried over to second-order partial differential equations (PDEs). Obviously it will not be possible to give a complete answer, due to the complexity of the study and the great variety of PDEs.

We consider the second-order PDEs of the form

$$A^{ij}u_{ij} - F(x^i, u, u_i) = 0 \tag{11}$$

for which at least one of the A^{ij} is nonzero and derive the point Lie symmetry conditions.

The symmetry condition (6) when applied to (11) gives

$$A^{ij}\eta_{ij}^{(2)} + \left(XA^{ij}\right)u_{ij} - X^{[1]}(F) = \lambda(A^{ij}u_{ij} - F) \tag{12}$$

that leads to

$$\begin{aligned}
&A^{ij}\eta_{ij} - \eta_{,i}g^{ij}F_{,u_j} - X(F) + \lambda F + 2A^{ij}\eta_{ui}u_j - A^{ij}\xi^a_{,ij}u_a - u_i\eta_u g^{ij}F_{,u_j} + \xi^k_{,i}u_k g^{ij}F_{,u_j} \\
&+ A^{ij}\eta_{uu}u_iu_j - 2A^{ij}\xi^k_{.,uj}u_iu_k + u_iu_k\xi^k_{,u}g^{ij}F_{,u_j} + A^{ij}\eta_u u_{ij} - 2A^{ij}\xi^k_{.,i}u_{jk} \\
&+ (\xi^kA^{ij}_{,k} + \eta A^{ij}_{,u})u_{ij} - \lambda A^{ij}u_{ij} - A^{ij}\left(u_{ij}u_a + u_iu_{ja} + u_{ia}u_j\right)\xi^a_{.,u} - u_iu_ju_aA^{ij}\xi^a_{uu} = 0.
\end{aligned} \tag{13}$$

We note that we cannot deduce the symmetry conditions before we select a specific form for the function F. However we may determine the conditions which are due to the second derivative of u because these terms do not involve F. This observation significantly reduces the complexity of the remaining symmetry condition. Following this observation we find the condition

$$\begin{aligned}
&A^{ij}\eta_u u_{ij} - A^{ij}(\xi^k_{.,i}u_{ja} + \xi^k_{.,j}u_{ik}) + (\xi^kA^{ij}_{,k} + \eta A^{ij}_{,u})u_{ij} - \lambda A^{ij}u_{ij} \\
&- A^{ij}\left(u_{ij}u_a + u_iu_{ja} + u_{ia}u_j\right)\xi^a_{.,u} - u_iu_ju_aA^{ij}\xi^a_{uu} = 0
\end{aligned}$$

which implies the equations

$$\begin{aligned}
A^{ij}\left(u_{ij}u_k + u_{jk}u_i + u_{ik}u_j\right)\xi^k_{.,u} &= 0, \\
A^{ij}\eta_u u_{ij} - A^{ij}(\xi^k_{.,i}u_{jk} + \xi^k_{.,j}u_{ik}) + (\xi^kA^{ij}_{,k} + \eta A^{ij}_{,u})u_{ij} - \lambda A^{ij}u_{ij} &= 0, \\
A^{ij}\xi^a_{uu} &= 0.
\end{aligned}$$

The first equation is written as

$$A^{ij}\xi^k_{.,u} + A^{kj}\xi^i_{.,u} + A^{ik}\xi^j_{.,u} = 0 \Leftrightarrow A^{(ij}\xi^{k)}_{.,u} = 0. \tag{14}$$

The second equation gives

$$A^{ij}\eta_u + \eta A^{ij}_{,u} + \xi^kA^{ij}_{,k} - A^{kj}\xi^i_{.,k} - A^{ik}\xi^j_{.,k} - \lambda A^{ij} = 0. \tag{15}$$

Therefore the last equation results in

$$\xi^k_{,uu} = 0. \tag{16}$$

It is straightforward to show that condition (14) implies $\xi^k_{.,u} = 0$ which is a well known result.

From the analysis so far we obtain that for all second-order PDEs of the form $A^{ij}u_{ij} - F(x^i, u, u_i) = 0$, for which at least one of the A^{ij} is nonzero, the coefficients $\xi^i_{.,u} = 0$ or $\xi^i = \xi^i(x^j)$.

Furthermore condition (16) is identically satisfied. We consider that A^{ij} is non-degenerate, furthermore the third symmetry condition (15) can be written as follows

$$L_{\xi^i\partial_i}A^{ij}=\lambda A^{ij}-(\eta A^{ij})_{,u}. \tag{17}$$

This condition implies that for all second-order PDEs of the form $A^{ij}u_{ij}-F(x^i,u,u_i)=0$ for which $A^{ij}{}_{,u}=0$, i.e. $A^{ij}=A^{ij}(x^i)$, the vector $\xi^i\partial_i$ is a CKV of the metric A^{ij} with conformal factor $(\lambda-\eta_u)(x)$.

Moreover, using that $\xi^i_{,u}=0$ when at least one of the $A_{ij}\neq 0$, the symmetry condition (13) is simplified as follows

$$\begin{aligned}&A^{ij}\eta_{ij}-\eta_{,i}A^{ij}F_{,u_j}-X(F)+\lambda F+2A^{ij}\eta_{ui}u_j-A^{ij}\xi^a_{,ij}u_a-u_i\eta_u A^{ij}F_{,u_j}\\&+\xi^k_{,i}u_kA^{ij}F_{,u_j}+A^{ij}\eta_{uu}u_iu_j+A^{ij}\eta_u u_{ij}-2A^{ij}\xi^k_{.,i}u_{jk}+\left(\xi^kA^{ij}_{,k}+\eta A^{ij}_{,u}\right)u_{ij}-\lambda A^{ij}u_{ij}=0,\end{aligned} \tag{18}$$

which together with the condition (17) are the complete set of conditions *for all* second-order PDEs of the form $A^{ij}u_{ij}-F(x^i,u,u_i)=0$ for which at least one of the $A_{ij}\neq 0$. This class of PDEs is quite general making the above result very useful.

In order to continue we need to assume that the function $F(x,u,u_i)$ is of a special form.

3.1. The Lie point symmetry conditions for a linear function $F(x,u,u_i)$

We consider the function $F(x,u,u_i)$ to be of the form

$$F(x,u,u_i)=B^k(x,u)u_k+f(x,u), \tag{19}$$

where $B^k(x,u)$ and $f(x,u)$ are arbitrary functions of their arguments. In this case the PDE is of the form

$$A^{ij}u_{ij}-B^k(x,u)u_k-f(x,u)=0. \tag{20}$$

The Lie symmetries of this type of PDEs have been studied previously by Ibragimov [1]. Assuming that at least one of the $A_{ij}\neq 0$ the Lie point symmetry conditions are (17) and (18).

Replacing $F(x,u,u_1)$ in (18) we obtain the following result [12]:

The Lie point symmetry conditions for the second-order PDEs of the form

$$A^{ij}u_{ij}-B^k(x,u)u_k-f(x,u)=0, \tag{21}$$

where at least one of the $A_{ij}\neq 0$, are

$$A^{ij}(a_{ij}u+b_{ij})-(a_{,i}u+b_{,i})B^i-\xi^kf_{,k}-auf_{,u}-bf_{,u}+\lambda f=0, \tag{22}$$

$$A^{ij}\xi^k_{,ij}-2A^{ik}a_{,i}+aB^k+auB^k_{,u}-\xi^k_{,i}B^i+\xi^iB^k_{,i}-\lambda B^k+bB^k_{,u}=0, \tag{23}$$

$$L_{\xi^i\partial_i}A^{ij}=(\lambda-a)A^{ij}-\eta A^{ij}{}_{,u}, \tag{24}$$

$$\eta=a(x^i)u+b(x^i), \tag{25}$$

$$\xi^k_{,u}=0\Leftrightarrow\xi^k(x^i). \tag{26}$$

From (24) we infer that for all second-order PDEs of the form $A^{ij}u_{ij}-B^k(x,u)u_k-f(x,u)=0$ for which $A^{ij}{}_{,u}=0$, the $\xi^i(x^j)$ is a CKV of the metric A^{ij}. Also in this case for the arbitrary function λ holds $\lambda=\lambda(x^i)$. This result establishes a relation between the Lie point symmetries of this type of PDEs with the collineations of the metric defined by the coefficients A_{ij}.

Furthermore in case[3] when $A^{tt}=A^{tx^i}=0$ and A^{ij} is a non-degenerate metric we obtain that

$$\xi^t_{,i}=0\Leftrightarrow\xi^t(t). \tag{27}$$

[3] The coordinates are t,x^i where $i=1,\ldots,n$.

These symmetry relations coincide with those given in [1]. Finally we note that equation (23) can be written as

$$A^{ij}\xi^{k}_{,ij} - 2A^{ik}a_{,i} + [\xi, B]^{k} + (a-\lambda)B^{k} + (au+b)B^{k}_{,u} = 0. \tag{28}$$

Let us see now some applications of these results.

4. Symmetries of the Poisson equation in a Riemannian space

The Lie symmetries of the Poisson equation

$$\Delta u - f\left(x^{i}, u\right) = 0, \tag{29}$$

where $\Delta = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^{i}}\left(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x^{j}}\right)$ is the Laplace operator of the metric g_{ij}, for $f = f(u)$ have been given in [1,19]. Here we generalize these results for the case $f = f\left(x^{i}, u\right)$.

The Lie symmetry conditions (22)–(26) for the Poisson equation (29) are

$$g^{ij}(a_{ij}u + b_{ij}) - (a_{,i}u + b_{,i})\Gamma^{i} - \xi^{k}f_{,k} - auf_{,u} - bf_{,u} + \lambda f = 0, \tag{30}$$

$$g^{ij}\xi^{k}_{,ij} - 2g^{ik}a_{,i} + a\Gamma^{k} - \xi^{k}_{,i}\Gamma^{i} + \xi^{i}\Gamma^{k}_{,i} - \lambda\Gamma^{k} = 0, \tag{31}$$

$$L_{\xi^{i}\partial_{i}}g_{ij} = (a-\lambda)g_{ij}, \tag{32}$$

$$\eta = a(x^{i})u + b(x^{i}), \quad \xi^{k}_{,u} = 0. \tag{33}$$

Equation (31) becomes (see [19])

$$g^{jk}L_{\xi}\Gamma^{i}_{.jk} = 2g^{ik}a_{,i}. \tag{34}$$

From (32), ξ^{i} is a CKV, then equations (34) give

$$\frac{2-n}{2}(a-\lambda)^{,i} = 2a^{,i} \to (a-\lambda)^{i} = \frac{4}{2-n}a^{,i}.$$

We define

$$\psi = \frac{2}{2-n}a + a_{0}, \tag{35}$$

where $\psi = \frac{1}{2}(a-\lambda)$ is the conformal factor of ξ^{i}, i.e. $L_{\xi}g_{ij} = 2\psi g_{ij}$. Furthermore, we have

$$\begin{aligned}(2-n)\lambda^{i} &= (2-n)a^{i} - 4a^{i},\\ (2-n)\lambda^{i} &= -(n+2)a^{i}.\end{aligned}$$

Finally, from (30), we have the constraint

$$g^{ij}a_{i;j}u + g^{ij}b_{;ij} - \xi^{k}f_{,k} - auf_{,u} + \lambda f - bf_{,u} = 0. \tag{36}$$

For $n=2$, it holds that $g^{jk}L_{\xi}\Gamma^{i}_{.jk} = 0$; this means that $a_{,i} = 0 \to a = a_{0}$. From (32), ξ^{i} is a CKV with conformal factor

$$2\psi = (a_{0} - \lambda) \tag{37}$$

and $\lambda = a_{0} - 2\psi$. Finally, from (30), we have the constraint

$$g^{ij}b_{;ij} - \xi^{k}f_{,k} - a_{0}uf_{,u} + (a_{0} - 2\psi)f - bf_{,u} = 0. \tag{38}$$

Hence for the Lie symmetries of the Poisson equation in a general Riemannian space we have the following theorem.

Theorem 1 *The Lie symmetries of the Poisson equation* (29) *are generated from the CKVs of the metric* g_{ij} *defining the Laplace operator, as follows*

a) for $n > 2$, *the Lie symmetry vector is*

$$X = \xi^i(x^k)\partial_i + \left(\frac{2-n}{2}\psi(x^k)u + a_0 u + b(x^k)\right)\partial_u, \tag{39}$$

where $\xi^i(x^k)$ *is a CKV with conformal factor* $\psi(x^k)$ *and the following condition holds*

$$\frac{2-n}{2}\Delta\psi u + g^{ij}b_{i;j} - \xi^k f_{,k} - \frac{2-n}{2}\psi u f_{,u} - \frac{2+n}{2}\psi f - b f_{,u} = 0. \tag{40}$$

b) for $n = 2$, *the Lie symmetry vector is*

$$X = \xi^i(x^k)\partial_i + (a_0 u + b(x^k))\partial_u, \tag{41}$$

where $\xi^i(x^k)$ *is a CKV with conformal factor* $\psi(x^k)$ *and the following condition holds*

$$g^{ij}b_{;ij} - \xi^k f_{,k} - a_0 u f_{,u} + (a_0 - 2\psi) f - b f_{,u} = 0.$$

5. Lie Symmetries of the Klein–Gordon equation and CKVs

In the special case, where $f(x^i, u) = -V(x^i)u$, the Poisson equation (29) is reduced to the Klein–Gordon equation

$$\Delta u + V(x^k)u = 0. \tag{42}$$

Therefore from theorem 1 for the Lie symmetries of (42) we have:

Theorem 2 *The Lie point symmetries of the Klein–Gordon equation* (42) *are generated from the elements of the conformal algebra of the metric* g_{ij} *defining the Laplace operator* Δ. *Equation* (42) *admits the Lie symmetry vector*

$$X = \xi^i(x^k)\partial_i + \left(\frac{2-n}{2}\psi(x^k)u + a_0 u + b(x^k)\right)\partial_u, \tag{43}$$

where $\xi^i(x^k)$ *is a CKV for the metric* g_{ij} *with conformal factor* $\psi(x^k)$, *the potential* $V(x^k)$ *satisfies the condition*

$$\xi^k V_{,k} + 2\psi V - \frac{2-n}{2}\Delta\psi = 0, \tag{44}$$

and the function $b(x^k)$ *is a solution of* (42).

Of interest are the cases where $V(x^k) = 0$ and $V(x^k) = \frac{n-2}{4(n-1)}R$, where R is the Ricci scalar of the metric g_{ij} which defines the operator Δ. In these cases, the Klein–Gordon equation (42) becomes the Laplace equation

$$\Delta u = 0 \tag{45}$$

and the conformal invariant Laplace equation

$$\bar{L}_g u = 0, \tag{46}$$

where $\bar{L}_g = \Delta + \frac{n-2}{4(n-1)}R$. Therefore, from theorem 2 for the Lie symmetries of equations (45) and (46) we have the following result

Theorem 3 *The generic form of the Lie point symmetry of Laplace equation* (45) *and of the conformal invariant Laplace equation* (46) *is the vector field* (43) *where for the Laplace equation* (45), *in a n-dimensional space with* $n > 2$, *the conformal factor* $\psi(x^k)$ *of the CKV* $\xi^i(x^k)$ *is a solution of* (45).

6. Reduction of Laplace equation in certain Riemannian spaces

From theorem 3 we have that the Lie symmetries of Laplace equation (45) in a Riemannian space are generated from the CKVs (not necessarily proper) whose conformal factor satisfies Laplace equation. This condition is satisfied trivially by the KVs ($\psi = 0$), the HV ($\psi_{;i} = 0$) and the sp.CKVs ($\psi_{;ij} = 0$). Therefore these vectors (which span a subalgebra of the conformal algebra) are among the Lie symmetries of Laplace equation. Concerning the proper CKVs it is not necessary that their conformal factor satisfies the Laplace equation, therefore they may not produce Lie symmetries for Laplace equation.

Furthermore, the special forms of the metric of a Riemannian space which admits a gradient KV/HV or a sp.CKV are well known in the literature. Therefore it is possible to study the application of the Lie symmetries of the Laplace equation in these Riemannian spaces. In the following, we study the reduction of Laplace equation in these general classes of Riemannian spaces and we also study the origin of type II hidden symmetries. We assume that the dimension n of the space is $n > 2$.

6.1. Reduction with a gradient KV/HV

We consider the $(1+n)$-dimensional metric g_{ij} with line element

$$ds^2 = dr^2 + r^{2K} h_{AB} dy^A dy^B, \quad h_{AB} = h_{AB}\left(y^C\right), \tag{47}$$

where h_{AB} is the metric of the n-dimensional space and $A, B, C = 1, \ldots, n$. For a general functional form of h_{AB} and when $K = 0$, the metric (47) admits the gradient KV ∂_r, however when $K = 1$, the latter admits the gradient HV $r\partial_r$ (see [20]).

For the space with line element (47) Laplace equation (45) takes the form

$$u_{,rr} + K\frac{n}{r}u_{,r} + \frac{1}{r^{2K}}\ {}_h\Delta u = 0, \tag{48}$$

where ${}_h\Delta u = h^{AB}\left(y^C\right)u_{,AB} - \Gamma^A\left(y^C\right)u_A$ is the Laplace operator with metric h_{AB}.

Laplace equation (48) admits extra Lie point symmetries when $K = 0, 1$. In particular when $K = 0$, the extra Lie point symmetry is the gradient KV $X_{KV} = \partial_r + \mu u\partial_u$ and when $K = 1$ the extra Lie point symmetry is the gradient HV $X_{HV} = r\partial_r + \mu u\partial_u$. We will study the reduction of equation (48) using the zero-order invariants of the symmetries X_{KV} and X_{HV}.

The zero-order invariants of X_{KV} are $\left\{y^A, e^{-\mu r}u\right\}$ and of X_{HV} are $\left\{y^A, r^{-\mu}u\right\}$. Hence by replacing in equation (48) we find the reduced equation

$${}_h\Delta w + \mu^2 w = 0, \tag{49}$$

where

$$u\left(r, y^A\right) = \left\{\begin{array}{ll} e^{\mu r} w\left(y^A\right) & \text{when} \quad K = 0 \\ r^{\mu} w\left(y^A\right) & \text{when} \quad K = 1 \end{array}\right\}.$$

Equation (49) is the linear Klein–Gordon equation in the space with metric h_{AB}. According to theorem 2 the Lie point symmetries of the reduced equation (49) follow from the CKVs of the metric h_{AB}.

The relation of the conformal algebra of the n metric h_{AB} and of the $1+n$ metric (47) have been studied in [21]. In particular the KVs of the metrics g_{ij} and h_{AB} are the same. Furthermore, for $K = 0$, the $1+n$ metric g_{ij} admits a HV if and only if the n metric admits one and if ${}_nH^A$ is the HV of the n metric then the HV of the $1+n$ metric is given by the expression

$${}_{1+n}H^{\mu} = z\delta^{\mu}_{z} +_n H^A \delta^{\mu}_{A}, \quad \mu = x, 1, \ldots, n. \tag{50}$$

However, for $K = 1$, the HV of the metric g_{ij} is independent from that of h_{AB}.

Finally, the metric (47) admits proper CKVs if and only if the n metric h_{AB} admits gradient CKVs. This is because (47) is conformally related with the decomposable metric

$$ds^{2\prime} = d\bar{r}^2 + h_{AB}\left(y^C\right) dy^A dy^B. \tag{51}$$

which admits CKVs if and only if the h_{AB} metric admits gradient CKVs.

The last implies, that Type II hidden symmetries are generated from the elements of the (proper) conformal algebra of the n-dimensional metric h_{AB} (for $K = 0, 1$) whose conformal factor is a solution of the Klein–Gordon equation (49), according to theorem 2. Furthermore when $K = 1$, the HV is a Type II hidden symmetry.

In the following we will study the origin of type II hidden symmetries in Riemannian spaces which admit a sp.CKV.

6.2. Reduction with a sp.CKV

It is known [22], that if an $n = m + 1$-dimensional ($n > 2$) Riemannian space admits a non null sp.CKVs then also admits a gradient HV and as many gradient KVs as the number of sp.CKVs. In these spaces there exists always a coordinate system in which the metric is written in the form

$$ds^2 = -dz^2 + dR^2 + R^2 f_{AB}\left(y^C\right) dy^A dy^B, \tag{52}$$

where $f_{AB}\left(y^C\right)$, $A, B, C, \ldots = 1, 2, \ldots, m-1$ is an $(m-1)$-dimensional metric. For a general metric f_{AB} the n-dimensional metric (52) admits a three-dimensional conformal algebra with elements

$$K_G = \partial_z, \quad H = z\partial_z + R\partial_R, \quad C_S = \frac{z^2 + R^2}{2}\partial_z + zR\partial_R,$$

where K_G is a gradient KV, H is a gradient HV and C_S is a sp.CKV with conformal factor $\psi_{C_S} = z$. In these coordinates Laplace equation (45) takes the form

$$-u_{zz} + u_{RR} + \frac{(m-1)}{R}u_R + \frac{1}{R^2}\, {}_f\Delta u_{AB} = 0. \tag{53}$$

From theorem 3, we have that the extra Lie point symmetries of (53) are the vectors

$$X^1 = K_G + \mu_G X_u, \quad X^2 = H + \mu_H X_u, \quad X^3 = C_S + 2pzX_u,$$

where $2p = \frac{1-m}{2}$. The nonzero commutators of the extra Lie point symmetries are

$$\left[X^1, X^2\right] = K_G, \quad \left[X^2, X^3\right] = X^3, \quad \left[X^1, X^3\right] = X^2 + 2pX_u.$$

The application of the Lie symmetries which are generated by the gradient KV and the gradient HV have been studied in section 6.1. However we would like to note that if we reduce the Laplace equation (53) by use of the Lie symmetry X^1, the reduced equation admits the inherited symmetry X^2 if and only if $\mu_G = 0$. Furthermore the reduction with the gradient HV leads to a PDE which does not admit inherited symmetries. The resulting type II hidden symmetries follow from the results of section 6.1.

Before we reduce (53) with the symmetry generated by the sp.CKV X^3, it is best to write the metric (52) in the coordinates $\left\{x, R, y^A\right\}$, where the variable x is defined by the relation $z = \sqrt{R\left(R - x^{-1}\right)}$. In the new variables the Lie symmetry X^3 becomes

$$X^3 = \sqrt{R\left(R - x^{-1}\right)}\left(R\partial_R + 2pu\partial_u\right). \tag{54}$$

The zero-order invariants of X^3 in the new coordinates are $\left\{x, y^A, R^{-2p}u\right\}$. We choose x, y^A to be the independent variables and $w = w\left(x, y^A\right)$ to be the dependent one; that is, the solution of the Laplace equation is in the form $u\left(x, R, y^A\right) = R^{2p}w\left(x, y^A\right)$. Replacing in (53) we find the reduced equation

$$x^2 w_{xx} + f^{AB}w_{AB} - \Gamma^A w_A - 2p\left(2p+1\right)w = 0. \tag{55}$$

For different values of the dimension m of the metric f_{AB}, equation (55) can be written in the following forms

$$\begin{aligned} x^2 w_{xx} + w_{yy} + \frac{1}{4}w &= 0, \text{ when } m = 2, && (56)\\ {}_{(m=3)}\bar{\Delta}w &= 0, \text{ when } m = 3, && (57)\\ {}_{(m\succeq 4)}\bar{\Delta}w - 2p\left(2p+1\right)V\left(\phi\right)w &= 0, \text{ when } m \succeq 4, && (58)\end{aligned}$$

where ${}_{(m=3)}\bar{\Delta}$ is the Laplace operator for the metric

$$d\bar{s}^2_{(m=3)} = \frac{1}{x^4}dx^2 + \frac{1}{x^2}f_{AB}dy^A dy^B. \tag{59}$$

The Laplace operator ${}_{(m\succeq 4)}\bar{\Delta}$ is defined by the metric

$$d\bar{s}^2_{(m\succeq 4)} = d\phi^2 + \frac{1}{V}f_{AB}dy^A dy^B, \tag{60}$$

where $V\left(\phi\right) = \dfrac{\left(2-m\right)^2}{\phi^2}$ and $d\phi = \dfrac{1}{xV}dx$.

By applying the Lie symmetry condition (6) for equation (56) we find that the generic Lie symmetry vector is

$$X = \xi^x\left(x,y\right)\partial_x + \xi^y\left(x,y\right)\partial_y + \left(a_0 w + b\right)\partial_w,$$

where

$$\begin{aligned} \xi^x\left(x,y\right) &= c_1 x + i\left(F_1\left(y + i\ln x\right) + F_2\left(y - i\ln x\right)\right),\\ \xi^y\left(x,y\right) &= F_2\left(y - i\ln x\right) - F_1\left(y + i\ln x\right). \end{aligned}$$

$\xi^i = \left(\xi^x, \xi^y\right)$ is the generic CKV of the two-dimensional metric $A^{ij} = diag\left(x^2, 1\right)$. Therefore all the proper CKVs of the two-dimensional metric A^{ij} generate type II hidden symmetries. Recall that the conformal algebra of a two-dimensional space is infinite-dimensional. The Lie point symmetry $x\partial_x$ is the inherited symmetry H.

Furthermore for equations (57) and (58) from theorems 3 and 2 we have that the type II hidden symmetries are generated by the proper CKVs of the metrics (59) and (60), respectively, with conformal factors such as the condition (43) holds. Finally equations (57) and (58) admit the inherited Lie point symmetry H.

7. Application of the reduction of the Laplace equation in Riemannian spaces

In section 6, we studied the reduction of Laplace equation and the origin of type II hidden symmetries in general Riemannian spaces which admit a gradient KV, a gradient HV and a sp.CKV. In this section we apply these general results in order to study the reduction of Laplace equation and the type II hidden symmetries in the case where the Laplace operator is defined by (a) the n-dimensional Minkowski spacetime M^n (b) the four-dimensional conformally flat Bianchi I spacetime which admits a gradient KV.

7.1. The $1+(n-1)$ wave equation

We consider the $1+(n-1)$ wave equation

$$-u_{,tt}+u_{,zz}+\delta^{AB}u_{,A}u_{,B}=0 \tag{61}$$

which is the Laplace equation in the n-dimensional Minkowski spacetime M^n with $n>3$.

The n-dimensional Minkowski spacetime admits a conformal group G_C of dimension $\dim G_c=\frac{1}{2}(n+1)(n+2)$. Furthermore, the conformal group admits the following subalgebras:

a. T^n, the translation group of n gradient KVs

$$K_G^1=\partial_t,\quad K_G^z=\partial_z,\quad K_G^A=\partial_{y^A};$$

b. $SO(n)$, the Lie group of $\frac{1}{2}n(n-1)$ rotations

$$X_R^{1A}=Y^\alpha\partial_t+t\partial_{Y^\alpha},\quad X_R^{\alpha\beta}=Y^\beta\partial_{Y^\alpha}-Y^\alpha\partial_{Y^\beta},$$

where $Y^\alpha=\left(z,y^A\right)$;

c. One gradient HV

$$H=t\partial_t+z\partial_z+y^A\partial_{y^A};$$

d. G_{spC}, the Lie group of n sp. CKVs

$$X_C^1=\frac{1}{2}\left(t^2+\sum_a\left(Y^\alpha\right)^2\right)\partial_t+t\sum_a\left(Y^\alpha\partial_{Y^\alpha}\right),$$

$$X_C^\alpha=tY^\alpha\partial_t+\frac{1}{2}\left(Y^\alpha+t^2-\sum_{\beta\neq\alpha}\left(Y^\beta\right)^2\right)\partial_{Y^\alpha}+Y^\alpha\sum_{\beta\neq\alpha}\left(Y^\beta\partial_{Y^\beta}\right).$$

Therefore from theorem 3 we have that the Lie symmetries of the wave equation (61) are

$$X_u=u\partial_u,\quad K_G^1,\quad K_G^z,\quad K_G^A,\quad X_R^{1A},\quad X_R^{AB},\quad H,\quad X_C^1-tX_u,\quad X_C^\alpha-Y^\alpha X_u$$

and that the nonzero commutators of the Lie symmetries are

$$\left[K_G^I,X_R^{IJ}\right]=-K_G^J,\quad\left[K_G^I,H\right]=K_G^I,\quad\left[K_G^I,X_C^I\right]=H-X_u,$$
$$\left[K_G^I,X_C^J\right]=X_R^{IJ},\quad\left[H,X_C^I\right]=X_C^I,\quad\left[X_R^{IJ},X_C^I\right]=X_C^J,$$

where $I=1,a$.

In order to apply the results of section 6 we reduce (61) by the use of the Lie point symmetries generated by the gradient KV K_G^z, the gradient HV H and the special CKV X_C^1.

Using the zero-order invariants of the Lie symmetries $K_G^z+\mu X_u$ in (61) the reduced equation is the linear Klein–Gordon equation in the M^{n-1} spacetime

$$-w_{,tt}+\delta^{AB}w_{,A}w_{,B}+\mu^2w=0, \tag{62}$$

where $u\left(t,z,y^A\right)=e^{\mu z}w\left(t,y^A\right)$.

For arbitrary constant μ, equation (62) admits the Lie point symmetries:

$$K_G^1,\quad K_G^A,\quad X_R^{1A},\quad X_R^{AB},\quad H,\quad w\partial_w,$$

which are the KVs and the HV of the M^{n-1}; these symmetries are inherited symmetries. However when $\mu=0$, equation (62) admits the extra Lie point symmetries

$$\bar{X}_C^1-\frac{1}{2}tw\partial_w,\quad \bar{X}_C^A-\frac{1}{2}y^Aw\partial_w,$$

where $\bar{X}_C^1$, $\bar{X}_C^A$ are the $(n-1)$ sp.CKVs of M^{n-1}; these symmetries are type II hidden symmetries.

We continue with the reduction of equation (61) by using the invariants of the Lie point symmetry $H+\mu X_u$. The wave equation (61) in hyperspherical coordinates (r,θ^A) is in the form of (48) with $K=1$ and ${}_h\Delta u$ is the Laplacian of the $n-1$ hyperbolic sphere S_h^{n-1} with line element

$$ds_h^2=d\theta_1^2+\cosh^2\theta_1\left(d\theta_2^2+\cosh^2\theta_2\left(d\theta_3^2+\cosh^2\theta_3(\dots)\right)\right). \tag{63}$$

Hence, from section 6 we have that the reduced equation is the linear Klein–Gordon equation in the hyperbolic sphere (63)

$${}_h\Delta w+\mu^2w=0, \tag{64}$$

where $u\left(r,\theta^A\right)=r^\mu w\left(\theta^A\right)$. The space S_h^{n-1} is a maximally symmetric space and admits a $\frac{1}{2}n(n+1)$-dimensional CKVs. In particular admits the $\frac{1}{2}n(n-1)$ KVs which form the $SO\left(n\right)$ Lie group and n proper gradient CKVs with conformal factor $\psi\left(\theta^A\right)$ such as

$${}_h\Delta\psi=\frac{2-n}{4(n-1)}R_h, \tag{65}$$

where R_h is the Ricci scalar of the S_h^{n-1} sphere which is a constant. Therefore from theorem 3 we have that for an arbitrary constant μ,equation (64) admits as Lie point symmetries the elements of the $SO\left(n\right)$ and the linear symmetry $w\partial_w$; these are inherited symmetries. However, when $\mu^2=\frac{n-2}{4(n-1)}R_h$ the Klein–Gordon equation (64) becomes the conformal invariant Laplace equation and admits $\frac{1}{2}n(n+1)$ Lie point symmetries, which span the conformal group of the space S_h^{n-1}. Therefore reduction with the gradient HV leads to the conformal invariant Laplace equation and the type II symmetries are generated by the proper CKVs of hyperbolic sphere S_h^{n-1}.

In order to reduce the wave equation (61) with the invariants of the Lie point symmetry $X_C^1-tX_u$ we use the coordinates $\left(t,R,\phi^K\right)$ and equation (61) becomes

$$u_{,.tt}-u_{,RR}-\frac{n-2}{R}u_R-\frac{1}{R^2}\,{}_h\bar{\Delta}u=0, \tag{66}$$

where ${}_h\bar{\Delta}u$ is the Laplacian of the $(n-2)$-dimensional hyperbolic sphere S_h^{n-2}. From section 6 we have that under the second coordinate transformation $t=\sqrt{R\left(R+\tau^{-1}\right)}$ the reduced equation is written in the simpler form

$$\begin{aligned}{}_{(m=3)}\bar{\Delta}w &= 0,\quad \text{when } n=4, \qquad (67)\\ {}_{(m\succeq4)}\bar{\Delta}w-2p\left(2p+1\right)\frac{\left(2-m\right)^2}{T^2}w &= 0,\quad \text{when } n>4, \qquad (68)\end{aligned}$$

where $u\left(\tau,R,\phi^K\right)=R^{2p}w\left(\tau,y^A\right)$ and ${}_{(m=3)}\bar{\Delta}w$, ${}_{(m\succeq4)}\bar{\Delta}w$ are the Laplace operators for the $(n-1)$-dimensional flat space $ds_{(n-1)}^2$ with line element

$$ds_{(n-1)}^2=dT^2-T^2ds_{h(n-2)}^2 \tag{69}$$

for the cases where $n=4$ and $n>4$ respectively. $ds^2_{h(n-2)}$ is the line element (63) of dimension $n-2$.

Therefore equation (67) admits ten extra Lie symmetries which form the conformal algebra of the flat space (69), whereas equation (68) admits as extra Lie point symmetries the gradient HV $T\partial_T$ and the rotation group $SO(n)$ of the metric (69) which is an inherited symmetry. Hence type II hidden symmetries we have only when $n=4$ and they are generated by KVs the translation group T^3 and the sp.CKVs of the flat space (69).

The reduction of the $1+3$ wave equation has been done previously in [23] and our results coincide with theirs when $n=4$. However, our approach is geometrical and we are able to recognize the type II hidden symmetries from the form of the metric which defines the Laplace operator without further calculations.

7.2. Bianchi I (diagonal) spacetime

The Bianchi I spacetime is defined as the four-dimensional spacetime which admits as KVs the translation group T^3 of the 3D Euclidian space [24]. The line element of the Bianchi I spacetime in the invariant basis where the KVs are $\{\partial_x, \partial_y, \partial_z\}$ is:

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2. \tag{70}$$

For general functions $A(t)$, $B(t)$, and $C(t)$ the KVs are non gradient. We restrict our considerations to the case where $C(t)=1$ so that the KV ∂_z is a gradient KV. In this case Laplace equation for the Bianchi I spacetime (70) becomes

$$-u_{,tt} + u_{,zz} + A^{-2}u_{,xx} + B^{-2}u_{,yy} - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)u_{,t} = 0. \tag{71}$$

From theorem 3, we have that the Laplace equation (71) admits as extra Lie point symmetries the T^3 Lie group. We consider the reduction of equation (71) with the symmetry

$$X_I = \partial_z + \mu u \partial_u.$$

From the application of the zero order invariance of X_I we have the reduced equation

$$_h\Delta w + \mu^2 w = 0, \tag{72}$$

where $_h\Delta w$ is the Laplace operator for the three-dimensional metric

$$ds^2_{(3)} = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2. \tag{73}$$

The three-dimensional spacetime (73) for general functions $A(t)$ and $B(t)$ admits the two-dimensional conformal algebra with elements the KVs $\{\partial_x, \partial_y\}$ which are inherited Lie point symmetries for the Klein–Gordon equation (72).

Moreover, when $A^2(t) = B^2(t)$, the spacetime (73) admits the KV $X_I^3 = y\partial_x - x\partial_y$ which is a Lie point symmetry of equation (72). However, X_I^3 is an inherited symmetry because it is also a KV of (70) and a Lie point symmetry of Laplace equation (71). Furthermore, when $A^2(t) = B^2(t) = t^2$ the space $ds^2_{(3)}$ admits the HV (73) which is a Lie point symmetry of equation (72) when $\mu = 0$. In this case H is a inherited symmetry because in that case metric (70) admits the HV H, which is also a Lie point symmetry for (71) with commutator $[X_I, H] = \partial_z$.

Finally, when $\left(A^2(t), B^2(t)\right) = \left(\sin^2 t, \cos^2 t\right)$ or $\left(A^2(t), B^2(t)\right) = \left(\sinh^2 t, \cosh^2 t\right)$ the three-dimensional spacetime (73) is a maximally symmetric spacetime, i.e. a space of constant non vanishing curvature, and admits six KVs and four gradient CKVs (for the conformal algebra

of the Bianchi I spacetime see [25]). Hence equation (72) for arbitrary μ admits six Lie point symmetries, the KVs. However, when $\mu = -\frac{1}{8}, R_{(3)}$ admits ten Lie point symmetries which are generated by the conformal algebra of (73), where $R_{(3)}$ is the Ricci scalar of the three-dimensional space $ds^2_{(3)}$. The type II hidden symmetries are the symmetries which are generated by the gradient CKVs. The KVs are inherited symmetries because they are also symmetries of equation (71).

8. Conclusion

In this work we have studied the connection between the Lie point symmetries of a general type of second-order PDEs and the collineations of the metric defined by the PDE itself. In particular we wrote the Lie symmetry conditions in a geometric form which include the Lie derivatives of geometric objects. Therefore, we proved that the Lie point symmetries of the PDEs of this family, which are linear in the first derivatives, are related with the elements of the conformal algebra of the metric defined by the PDE modulo a constraint condition depending again on the PDE. Important elements in this family are the Poisson equation, the Klein–Gordon equation and the Laplace equation. To these equations we have derived the Lie point symmetries for various classes of Riemannian spaces. In particular we have studied the type II hidden symmetries of the wave equation in Minkowski space and in Bianchi I spacetimes. The methodology we followed is geometric and can be applied to other types of PDEs as well as to this type of PDEs in other Riemannian spaces.

Acknowledgments

AP acknowledges financial support of INFN.

References

[1] Ibragimov N H 1985 *Transformation Groups Applied to Mathematical Physics* (Dordrecht: D. Reidel Publishing Co)
[2] Stephani H 1989 *Differential Equations: Their Solutions Using Symmetry* (New York: Cambridge University Press)
[3] Bluman G W and Kumei S 1989 *Symmetries of Differential Equations* (New York: Springer-Verlag)
[4] Prince G E and Crampin M 1984 *Gen. Relativity Gravitation* **16** 921
[5] Aminova A V and Aminov N A-M 2006 *Sb. Math.* **197** 951
[6] Aminova A V and Aminov N A-M 2010 *Sb. Math.* **201** 631
[7] Tsamparlis M, Paliathanasis A 2010 *Nonlinear Dynam.* **62** 203
[8] Tsamparlis M and Paliathanasis A 2010 *Gen. Relativity Gravitation* **42** 2957
[9] Tsamparlis M and Paliathanasis A 2011 *Gen. Relativity Gravitation* **43** 1861
[10] Barnes A 1993 *Classical Quantum Gravity* **10** 1139
[11] Bozhkov Y and Freire I L *J. Differential Equations* **249** (2010) 872
[12] Paliathanasis A and Tsamparlis M 2012 *J. Geom. Phys.* **62** 2443
[13] Paliathanasis A and Tsamparlis M 2014 *J. Geom. Phys.* **76** 107
[14] Paliathanasis A and Tsamparlis M 2014 *Int. J. Geom. Methods Mod. Phys.* **11** 1450037
[15] Tsamparlis M, Paliathanasis A and Qadir A 2015 *Int. J. Geom. Methods Mod. Phys.* **12** 1550003
[16] Daftardar V and Dadhich N 1994 *Gen. Relativity Gravitation* **26** 859
[17] Govinger K S 2001 *J. Math. Anal. Appl.* **258** 720
[18] Katzin G H, Levine J and Davis W R 1969 *J. Math. Phys.* **10** 617
[19] Bozhkov Y and Freire I L 2010 *J. Differential Equations* **249** 872
[20] Colley A A and Tupper B O J 1989 *J. Math. Phys.* **30** 2616
[21] Tsamparlis M, Nikolopoulos D and Apostolopoulos P S 1998 *Classical Quantum Gravity* **15** 2909
[22] Hall G S and Roy I M 1997 *Gen. Relativity Gravitation* **29** 827
[23] Abraham-Shrauner B, Govinder K S and Arrigo D J 2006 *J. Phys. A: Math Gen* **39** 5739
[24] Rayan M P Jr and Shepley L C 1975 *Homogeneous Relativistic Cosmologies* (Princeton: Princeton University Press)
[25] Tsamparlis M and Apostolopoulos P S 2000 *J. Math. Phys.* **41** 7573